Renormalisation group treatment of finite size scaling with ε expansion

1985 J. Phys. A: Math. Gen. 18 L319

(http://iopscience.iop.org/0305-4470/18/6/011)

J. Phys. A: Math. Gen. 18 (1985) L319–L323. Printed in Great Britain

LETTER TO THE EDITOR

Renormalisation group treatment of finite size scaling with ε expansion

A M Nemirovsky and Karl F Freed

The James Franck Institute, The University of Chicago, Chicago, Illinois 60637, USA

Received 12 December 1984

Abstract. The use of renormalisation group techniques away from the critical point is shown to enable the calculation of finite size scaling corrections to scaling functions with ε-expansion methods. We consider the N-vector ϕ^4 theory for a layered geometry with periodic boundary conditions and evaluate the correlation function, susceptibility, correlation lengths and shift in the critical temperature to order ε.

Finite size effects on phase transitions are of considerable current interest both theoretically and experimentally. (A good review of the subject is given by Barber (1983).) Different geometries can be considered, e.g., a completely finite system in d dimensions or a system infinite in one (or $d-1$) dimensions but finite in the other $d-1$ (or one) dimensions, etc. Boundary conditions (periodic, antiperiodic, free surfaces, etc) must be specified for a given geometry. In this work we illustrate the general theory with a d-dimensional layered geometry which is infinite in $d-1$ dimensions, is of thickness L in the remaining dimension, and has periodic boundary conditions.

The finite size scaling hypothesis (Fisher 1971, Fisher and Barber 1972) is widely used to extrapolate results from finite or partially finite systems to the thermodynamic limit. This hypothesis states that in the vicinity of the bulk critical temperature T_c^∞, the behaviour of the finite system can be described in terms of the dimensionless scaling variable $y = Lt^\nu$, where $t = (T - T_c^\infty)/T_c^\infty$ is the reduced temperature, ν is the usual d-dimensional critical exponent, i.e. $\xi \sim t^{-\nu}$ where ξ is the correlation length of a system of infinite extent, and L is the characteristic length scale for the finite size of the system. For a layered geometry L is just the single layer thickness. Among other effects the finite size produces a shift in the critical (or pseudocritical) temperature from T_c^∞ to T_c^L. The 'shift' exponent λ describes the L dependence of this shift by

$$T_c^\infty - T_c^L \sim L^{-\lambda}, \qquad L \to \infty.$$

The ε-expansion method is one of the most powerful renormalisation group techniques used to study critical phenomena in infinite systems (see e.g. Amit 1978). It has more recently been extended to semi-infinite systems (Symanzik 1981, Diehl and Dietrich 1981, Nemirovsky and Freed 1985 and references therein). Hence, it is important to generalise this technique for finite size systems. However, it has been argued (Brézin 1982) that finite size scaling functions are well defined for $\varepsilon > 0$ but become singular as $\varepsilon \to 0^+$, thereby apparently precluding the possibility of evaluating the finite size scaling functions by ε-expansion methods.

0305-4470/85/060319+05$02.25

In this letter we show that, as long as $Lt^{\nu} \gg 1$, the ε expansion is well defined. Calculations are performed for the N-vector ϕ^4 field theory with a layered geometry and periodic boundary conditions. We explicitly evaluate the correlation function, susceptibility, and parallel and perpendicular correlation lengths to first order in ε. The scaling hypothesis is satisfied for $d \leqslant 4$ and scaling functions are given to $O(\varepsilon)$. Also, it is found that, away from the critical region and in the $L \to \infty$ limit, finite size corrections become exponentially small as suggested by previous calculations on various models (Ferdinand and Fisher 1969, Barber and Fisher 1973, Barber 1973, 1977). In addition, the critical exponents are those of an infinite d-dimensional system, as long as $Lt^{\nu} \gg 1$. When this condition is violated the perturbative series expansion breaks down, becoming an expansion in a large dimensionless parameter $(Lt^{\nu})^{-1}$. The shift in the critical temperature to $O(\varepsilon)$ is calculated and yields $\lambda = 2$.

Bounded field theories are also of current interest in areas such as particle physics (Bernard 1974, Weinberg 1974, Dolan and Jackiw 1974) and general relativity (Toms 1980, Birrell and Ford 1980), and some results obtained by these authors have been utilised in this work.

We use an N-component scalar ϕ^4 theory for a d-dimensional layered system of infinite extent in $d-1$ dimensions and of thickness L along the remaining dth direction. Periodic boundary conditions are applied in the dth dimension, i.e., the local order parameter $\phi(\boldsymbol{\rho}, z)$ satisfies

$$\phi(\boldsymbol{\rho}, z) = \phi(\boldsymbol{\rho}, z+L), \tag{1}$$

where $\boldsymbol{\rho}$ is a $(d-1)$-dimensional vector perpendicular to the thickness. The Landau free energy functional for the model is given by

$$F\{\phi\} = \int_0^L \mathrm{d}z \int \mathrm{d}^{d-1}\rho \left(\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}t_0\phi^2 + \frac{g_0}{4!}(\phi^2)^2\right), \tag{2}$$

where the parameters t_0 and g_0 are the bare reduced temperature and coupling constant, respectively.

Since $\phi(\boldsymbol{\rho}, z)$ is periodic in the interval $[0, L]$, it can be expanded in a Fourier series

$$\phi(\boldsymbol{\rho}, z) = L^{-1} \sum_{j=-\infty}^{\infty} \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}} \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\rho} + \mathrm{i}\kappa_j z)\phi_j(\boldsymbol{k}), \tag{3}$$

with $\boldsymbol{k}$ the Fourier variable conjugate to $\boldsymbol{\rho}$, and $\kappa_j = 2\pi j/L$. Feynman rules for the perturbation expansion are the usual ones for the full space (Amit 1978), apart from the replacement (Bernard 1974)

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \to \frac{1}{L}\sum_{j=-\infty}^{\infty} \int \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}}, \tag{4a}$$

$$k_z \to \kappa_j, \qquad \kappa_j = 2\pi j/L, \tag{4b}$$

$$(2\pi)^d\delta^d(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \ldots) \to L\delta_{\kappa_{j_1}+\kappa_{j_2}+\ldots}(2\pi)^{d-1}\delta^{d-1}(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \ldots). \tag{4c}$$

These rules are used to evaluate the connected two-point function $G^{(2)}$ (correlation function) to first order in g_0 as

$$\begin{aligned} G^{(2)}(k, \kappa_j, t) = {} & (k^2 + \kappa_j^2 + t_0)^{-1} - \bar{g}_0[(N+2)/12]\Gamma(d/2)\Gamma(1-d/2)(k^2+\kappa_j^2+t_0)^{-2} \\ & - \bar{g}_0[(N+2)/3](2\pi)^{d-2}\pi^{-1/2}\Gamma(\tfrac{3}{2}-d/2)\Gamma(d/2)\sin(\tfrac{3}{2}-d/2)\pi L^{2-d} \\ & \times f_{3/2-d/2}(Lt_0^{1/2}/2\pi)(k^2+\kappa_j^2+t_0)^{-2} + O(\bar{g}_0^2), \end{aligned} \tag{5}$$

where $\bar{g}_0 = g_0 S_d/(2\pi)^d$, $S_d = 2\pi^{d/2}/\Gamma(d/2)$, and we have used (Birrell and Ford 1980)

$$\sum_{j=-\infty}^{\infty} (j^2+a^2)^{-\alpha} = \pi^{1/2} a^{1-2\alpha} \frac{\Gamma(\alpha-\frac{1}{2})}{\Gamma(\alpha)} + 4\sin(\pi\alpha) f_\alpha(a) \tag{6a}$$

with

$$f_\alpha(a) = \int_a^\infty \frac{(u^2-a^2)^{-\alpha}}{e^{2\pi u}-1}\,du. \tag{6b}$$

The divergence of the correlation function $G^{(2)}$ in equation (5) is the same as that of the full-space theory. In fact, it has been shown that, at least up to two loops, the theory we are considering here is rendered finite by the full-space normalisation constants (Kislinger and Morley 1976), i.e. the divergences are the same as those for the infinite system.

The renormalised field, coupling constant and temperature are given as usual by

$$\phi_R = Z_\phi^{-1/2}\phi, \qquad g = Z_g^{-1}\mu^{-\varepsilon/2}\bar{g}_0, \qquad t = Z_t^{-1} t_0, \tag{7}$$

with μ a parameter having dimensions of temperature that is used to define a dimensionless coupling constant. The functions Z are given by (Amit 1978)

$$Z_\phi = 1 + O(g^2), \qquad Z_g = 1 + \frac{N+8}{6\varepsilon} g + O(g^2), \qquad Z_t = 1 + \frac{N+2}{6\varepsilon} g + O(g^2). \tag{8}$$

These renormalisation constants enable us to define the renormalised correlation function by

$$G_R^{(2)}(t, g) = Z_\phi^{-1} G^{(2)}(Z_t t, Z_g g \mu^{\varepsilon/2}). \tag{9}$$

Combining (7)-(9) with (5) gives

$$\begin{aligned} G_R^{(2)} = {} & (k^2+\kappa_j^2+t)^{-1} - g[(N+2)/12] t (k^2+\kappa_j^2+t)^{-2} \ln(t/\mu) \\ & - g[(N+2)/3] 2(2\pi/t^{1/2}L)^2 t (k^2\kappa_j^2+t)^{-2} f_{-1/2}(t^{1/2}L/2\pi) + O(g^2), \end{aligned} \tag{10}$$

where $f_{-1/2}(a)$ is given by (6b) with $\alpha = -\frac{1}{2}$.

The susceptibility χ and correlation lengths $\zeta_\parallel$ and $\zeta_\perp$ are readily evaluated from $G_R^{(2)}$ through

$$\chi(t, L) = \int_0^L dz \int d^{d-1}\rho\, G_R^{(2)}(\boldsymbol{\rho}, z) = G_R^{(2)}(\boldsymbol{k}=0, \kappa_j=0), \tag{11a}$$

$$\zeta_\parallel^2 = \frac{\int_0^L dz \int d^{d-1}\rho\, |\boldsymbol{\rho}|^2 G_R^{(2)}(\boldsymbol{\rho}, z)}{\int_0^L dz \int d^{d-1}\rho\, G_R^{(2)}(\boldsymbol{\rho}, z)}, \tag{11b}$$

$$\zeta_\perp^2 = \frac{\int_0^L dz\, z^2 \int d^{d-1}\rho\, G_R^{(2)}(\boldsymbol{\rho}, z)}{\int_0^L dz \int d^{d-1}\rho\, G_R^{(2)}(\boldsymbol{\rho}, z)}. \tag{11c}$$

At the fixed point $g^* = [6/(N+8)]\varepsilon + O(\varepsilon^2)$, the renormalisation-group equation implies the scaling forms

$$\chi = \frac{1}{\mu}\left(\frac{t}{\mu}\right)^{-\gamma} K(y), \qquad \zeta_{\parallel,\perp}^2 = \frac{1}{\mu}\left(\frac{t}{\mu}\right)^{-2\nu} H_{\parallel,\perp}(y), \tag{12a, b}$$

where the critical exponents γ and ν are the d-dimensional bulk ones, i.e.,

$$\gamma = 1 + \frac{1}{2}\frac{N+2}{N+8}\varepsilon + O(\varepsilon^2), \qquad \nu = \frac{1}{2} + \frac{1}{4}\frac{N+2}{N+8}\varepsilon + O(\varepsilon^2). \tag{13a, b}$$

K, $H_\parallel$ and $H_\perp$ are dimensionless functions of the dimensionless variables $y = Lt^\nu$. After some algebra, and making use of (10) and (11), they are found to be

$$K(y) = 1 - \varepsilon[(N+2)/(N+8)]4(2\pi/y)^2 f_{-1/2}(y/2\pi) + O(\varepsilon^2), \tag{14a}$$

$$H_\parallel(y) = 2(d-1)\{1 - \varepsilon[(N+2)/(N+8)]4(2\pi/y)^2 f_{-1/2}(y/2\pi) + O(\varepsilon^2)\}, \tag{14b}$$

$$H_\perp(y) = 2[1 + (y/2)^2 - (y/2)\coth(y/2)] + \varepsilon[(N+2)/(N+8)],$$

$$4(2\pi/y)^2 f_{-1/2}(y/2\pi)[-2 + (y/2)^2 \operatorname{cosech}^2(y/2) + (y/2)\coth(y/2)] + O(\varepsilon^2). \tag{14c}$$

The function $f_{-1/2}(y/2\pi)$ defined by (6b) has the asymptotic limits (Birrell and Ford 1980)

$$f_{-1/2}(y/2\pi) = \tfrac{1}{2}(2\pi)^{-3/2} y^{1/2}\, e^{-y}, \qquad y \to \infty, \tag{15a}$$

$$f_{-1/2}(0) = \tfrac{1}{24}. \tag{15b}$$

Equation (15a) implies that away from criticality ($t > 0$) and in the limit of very large thickness ($L \to \infty$), finite size corrections to the correlation function (equation (10)), susceptibility and correlation lengths (equations (12) to (14)) are exponentially small.

Combining (15b) with (10) or (12)-(14) indicates that as $Lt^\nu \to 0$ the perturbative expansion is one in the large dimensionless parameter $(Lt^\nu)^{-1}$, and the first-order contributions become arbitrarily larger than the zeroth-order ones. As long as $Lt^\nu \geqslant 1$ the series are well behaved. However, had we considered a critical theory from the beginning, i.e., calculation and renormalisation at $t = 0$, as is usually done for infinitely extended systems, we would not have obtained meaningful results.

Finally, using the Schwinger–Dyson equation together with (10), the shift in the critical temperature to $O(\varepsilon)$ is calculated to be

$$t_c = -\frac{\varepsilon}{6}\frac{N+2}{N+8}\left(\frac{2\pi}{L}\right)^2 + O(\varepsilon^2). \tag{16}$$

Since the theory discussed in this letter is ill defined below $t \sim L^{-1/\nu}$, the result of equation (16), which predicts a $t_c < 0$, should be taken with some caution. This point has been extensively discussed (Weinberg 1974) in the context of finite temperature field theory. We provide a more rigorous treatment of (16) in a forthcoming work giving an expanded discussion of finite size effects for a d-dimensional layered geometry with periodic and with antiperiodic boundary conditions.

This research is supported by NSF grant DMR 83-18560 and has benefited from the use of MRL(NSF) facilities at the University of Chicago. We are grateful to Professor K Binder for his encouragement and Professors J D Bjorken and J L Rosner for useful discussions.

References

Amit D J 1978 *Field Theory, the Renormalization Group and Critical Phenomena* (New York: McGraw-Hill)
Barber M N 1973 *Aust. J. Phys.* **26** 483
—— 1977 *J. Phys. A: Math. Gen.* **10** 1335
—— 1983 *Finite-Size Scaling* ed C Domb and J L Lebowitz (New York: Academic) vol 8
Barber M N and Fisher M E 1973 *Ann. Phys.* **77** 1

Bernard C W 1974 *Phys. Rev.* D **9** 3312
Birrell N D and Ford L H 1980 *Phys. Rev.* D **22** 330
Brézin E 1982 *J. Physique* **43** 15
Diehl H W and Dietrich S 1981 *Z. Phys.* B **42** 65
Dolan L and Jackiw R 1974 *Phys. Rev.* D **9** 3320
Ferdinand A E and Fisher M E 1969 *Phys. Rev.* **185** 832
Fisher M E 1971 *Critical Phenomena, Proc. 51st Enrico Fermi Summer School, Varenna, Italy* ed M S Green (New York: Academic)
Fisher M E and Barber M N 1972 *Phys. Rev. Lett.* **28** 1516
Kislinger M B and Morley P D 1976 *Phys. Rev.* D **13** 2771
Nemirovsky A M and Freed K F 1985 *Phys. Rev.* B in press
Symanzik K 1981 *Nucl. Phys.* B **190** [F53] 1
Toms D J 1980 *Phys. Rev.* D **21** 928
Weinberg S 1974 *Phys. Rev.* D **9** 3357